ON THE PROBLEM OF THE MINIMUM SUM OF THE DISTANCES OF A POINT FROM GIVEN POINTS.*

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CONTENTS.

		PAGE
Art. 1.	*Introductory note.* History of the problem	33
" 2.	*Differential equation between the distances of a variable point P from n fixed points in an (n − 1)-dimensional space*	34
" 3.	Specialization for the plane and for ordinary space	37
" 4.	*Minimum of the sum of the distances of the point P from the points* $e_1, e_2, \ldots e_n$. First principal formula (extension of Steiner's formula)	38
" 5.	Specialization for the plane and for ordinary space	38
" 6.	Other particular cases	40
" 7.	Continuation	40
" 8.	*Geometrical interpretations of the general result.* Second principal formula (transformation of the first)	41
" 9.	Third principal formula (extension of Sturm's formula)	42
" 10.	Exposition of the connection between Steiner's and Sturm's formulæ (transformation of the third principal formula into the second)	44
" 11.	Particular cases	45
" 12.	Continuation	45
" 13.	Interpretation of the general minimum point as centroid	46
" 14.	*Remarks concerning the construction of the minimum point.* —The case of n points in $(n-1)$-dimensional space	46
" 15.	The case of n points in any space. First method	47
" 16.	Second method	48
" 17.	The case of n points in a straight line	48
" 18.	Particular case	50
" 19.	Transition of the minimum point into a vector	50
" 20.	Application to a practical case	50
" 21.	*Application to a problem in mechanics*	51
" 22.	*Concluding remark*	52

1. *Introductory Note.*—Although repeatedly treated before,† the problem of determining a point such that the sum of its distances from a number of given points shall be a minimum still offers room for discussion in two respects.

First, the question as to the best method of investigation cannot yet be regarded as settled. The ordinary method of the differential calculus not only leads to unnecessary complications owing to the use of co-ordinates not essentially connected with the problem, but also necessitates the exclusion of negative distances, a serious difficulty that does not seem to have been

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† See, for instance, the references to the literature of the problem given by STURM in *Crelle's Journal für Mathematik*, vol. 97, p. 51.

overcome.* The methods of elementary geometry, on the other hand, always comprise only a comparatively small class of problems, and even the simplest solutions, like those for the sum of the distances from three or four points, give decidedly the impression of being nothing but special artifices. It is to the researches of Sturm that we owe the important result that all these particular solutions can be regarded as special cases of a general method applicable to n points. Nevertheless, there seems to be but little prospect of applying these purely geometrical methods successfully to those generalized problems that are mentioned by Steiner in the brief notice of his paper read before the Berlin Academy.† Indeed, the character of these generalizations which relate to sums, not of mere distances, but of multiples of powers of the distances, would seem to point in the direction of analytical methods. The single result for this kind of problems given by Steiner himself‡ bears so openly the mark of an analytical origin that an analytical investigation of the generalized problem appears to offer the best chances of success in uncovering the "easy geometrical considerations" that sufficed Steiner, as is implied in the notice of his paper, to solve the most general case of the problem.

Secondly, our problem lends itself to a series of generalizations which do not appear as yet to have been considered.

In what follows the problem is treated by means of the simplest methods of Grassmann's geometrical calculus (Ausdehnungslehre).

2. *Differential equation between the distances of a variable point P from n fixed points in an* $(n-1)$*-dimensional space.* Let P be a variable point derived from n fixed points e_1, e_2, . . . e_n by means of the numbers α_1, α_2, . . . α_n, so that

$$P = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n; \quad \alpha_1 + \alpha_2 + \ldots + \alpha_n = 1;$$
$$(e_1 e_2 \ldots e_n) = 1. \qquad (1)$$

Then the outer product $(e_1 e_2 \ldots e_n)$ represents the $(n-1)!$-fold volume of the $(n-1)$-dimensional n-gon formed by the points e_1, e_2, . . . e_n, i.e., of the nth term in the series of configurations beginning with the terms: point, rectilinear segment, triangle, tetrahedron, pentahedroid (Fünfzell), etc. The coefficients α_1, α_2, . . . α_n represent the $(n-1)!$-fold volumes of the $(n-1)$-dimensional n-gons $Pe_2e_3 \ldots e_n$, $Pe_1e_3 \ldots e_n$, . . .

* See STURM, *l. c.*

† *Monatsberichte der Akademie der Wissenschaften*, Berlin, 1837, p 144; STEINER, *Gesammelte Werke*, vol. II. p. 95.

‡ *Crelle's Journal für Mathematik*, vol. 18, p. 362; STEINER, *Gesam. Werke*, vol. II. p. 17.

$Pe_1e_2 \ldots e_{n-1}$. For we find from (1) by outer multiplication, say by $(e_2e_3 \ldots e_n)$,

$$(Pe_2e_3 \ldots e_n) = \alpha_1(e_1e_2 \ldots e_n) = \alpha_1.$$

Now $(Pe_1)=\alpha_2(e_2e_1) + \alpha_3(e_3e_1) + \ldots + \alpha_n(e_ne_1)$; and forming the inner square:

$$(Pe_1)^2=\Sigma\alpha_r^2(e_re_1)^2+\Sigma\alpha_r\alpha_s[e_re_1 \mid e_se_1];\ r \lesssim s;\ r, s=2, 3, \ldots n.$$

On the other hand, denoting by p_1, a_{1r}, a_{1s} the numerical values of the rotors or point-vectors (Linienteile) (Pe_1), (e_re_1), (e_se_1), we have, with the same conditions for r and s,

$$p_1^2 = \Sigma\alpha_r^2a_{1r}^2 + \Sigma\alpha_r\alpha_sa_{1r}a_{1s}\cos\phi_{rs},$$

where ϕ_{rs} is the angle between the edges a_{1r} and a_{1s}. Replacing the expression $a_{1r}a_{1s}\cos\phi_{rs}$ by $\frac{1}{2}(a_{1r}^2 + a_{1s}^2 - a_{rs}^2)$, we find

$$p_1^2 = \Sigma\alpha_r^2a_{1r}^2 + \tfrac{1}{2}\Sigma\alpha_r\alpha_s(a_{1r}^2 + a_{1s}^2 - a_{rs}^2), \qquad (2)$$

or, writing separately the terms in which $r, s = n$,

$$\begin{aligned} p_1^2 &= \alpha_n^2a_{1n}^2 + \Sigma\alpha_r^2a_{1r}^2 + \Sigma\alpha_r\alpha_n(a_{1r}^2 + a_{1n}^2 - a_{rn}^2) \\ &\quad + \tfrac{1}{2}\Sigma\alpha_r\alpha_s(a_{1r}^2 + a_{1s}^2 - a_{rs}^2);\ r, s = 2, 3, \ldots (n-1). \end{aligned} \qquad (3)$$

According to equation (1), the position of P is determined by the co-ordinates $\alpha_1, \alpha_2, \ldots \alpha_n$, of which $(n-1)$, say $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$, can be regarded as independent. As, by (1),

$$\alpha_n = 1 - \alpha_1 - \alpha_2 - \ldots - \alpha_{n-1},$$

we have

$$\frac{d\alpha_n}{d\alpha_1} = \frac{d\alpha_n}{d\alpha_2} = \ldots = \frac{d\alpha_n}{d\alpha_{n-1}} = -1.$$

Hence (2) gives:

$$\begin{aligned} \frac{d(p_1^2)}{d\alpha_1} &= -2\alpha_na_{1n}^2 - \Sigma\alpha_r(a_{1r}^2 + a_{1n}^2 - a_{rn}^2); \\ & r = 2, 3, \ldots (n-1). \end{aligned} \qquad (4)$$

Let k denote any one of the subscripts $2, 3, \ldots (n-1)$; then we have by analogy with (2)

$$\begin{aligned} p_k^2 &= \Sigma\alpha_r^2a_{kr}^2 + \tfrac{1}{2}\Sigma\alpha_r\alpha_s(a_{kr}^2 + a_{ks}^2 - a_{rs}^2); \\ r &\gtrless s;\ r, s = 1, 2, \ldots (k-1), (k+1), \ldots n. \end{aligned} \qquad (5)$$

or, writing separately the terms in which r, $s = 1$, n,

$$p_k^2 = \alpha_1^2 a_{k1}^2 + \alpha_n^2 a_{kn}^2 + \Sigma \alpha_r^2 a_{kr}^2 + \alpha_1 \alpha_n (a_{k1}^2 + a_{kn}^2 - a_{1n}^2) + \Sigma \alpha_r \alpha_1 (a_{kr}^2 + a_{k1}^2 - a_{r1}^2) + \Sigma \alpha_r \alpha_n (a_{kr}^2 + a_{kn}^2 - a_{rn}^2) + \tfrac{1}{2} \Sigma \alpha_r \alpha_s (a_{kr}^2 + a_{ks}^2 - a_{rs}^2); \quad r, s = 2, 3, \ldots (k-1), (k+1), \ldots (n-1). \tag{6}$$

Hence

$$\frac{d(p_k^2)}{d\alpha_1} = \alpha_n (a_{k1}^2 - a_{n1}^2 - a_{kn}^2) + \alpha_1 (a_{k1}^2 + a_{n1}^2 - a_{kn}^2) + \Sigma \alpha_r (a_{k1}^2 - a_{r1}^2 + a_{rn}^2 - a_{kn}^2); \quad r = 2, 3, \ldots (k-1), (k+1), \ldots (n-1). \tag{7}$$

Interchanging the subscripts 1 and n in (3), we find by differentiation

$$\frac{d(p_n^2)}{d\alpha_1} = 2\alpha_1 a_{n1}^2 + \Sigma \alpha_r (a_{n1}^2 + a_{nr}^2 - a_{1r}^2); \quad r = 2, 3, \ldots (n-1). \tag{8}$$

Let us now substitute in (7) the values 2, 3, $\ldots$ $(n-1)$ successively for k and add all the resulting equations, together with the equations (4) and (8), after multiplying them respectively by α_k, α_1, α_n; in the resulting equation,

$$\alpha_1 \frac{d(p_1^2)}{d\alpha_1} + \Sigma \alpha_k \frac{d(p_k^2)}{d\alpha_1} + \alpha_n \frac{d(p_n^2)}{d\alpha_1} = -2\alpha_n \alpha_1 a_{1n}^2 - \alpha_1 \Sigma \alpha_r (a_{1r}^2 + a_{1n}^2 - a_{rn}^2) + \alpha_n \Sigma \alpha_k (a_{k1}^2 - a_{n1}^2 - a_{kn}^2) + \alpha_1 \Sigma \alpha_k (a_{k1}^2 + a_{n1}^2 - a_{kn}^2) + \Sigma \alpha_k \alpha_r (a_{k1}^2 - a_{r1}^2 + a_{rn}^2 - a_{kn}^2) + 2\alpha_1 \alpha_n a_{n1}^2 + \alpha_n \Sigma \alpha_r (a_{n1}^2 + a_{nr}^2 - a_{1r}^2); \quad r, k = 2, 3, \ldots (n-1),$$

the first and sixth terms in the right-hand member cancel, as do also the second and fourth, the third and seventh, while the fifth vanishes identically, since any two terms of the forms

$$\alpha_k \alpha_r (a_{k1}^2 - a_{r1}^2 + a_{rn}^2 - a_{kn}^2), \qquad \alpha_r \alpha_k (a_{r1}^2 - a_{k1}^2 + a_{kn}^2 - a_{rn}^2)$$

have the sum zero. We have therefore

$$\Sigma \alpha_k \frac{d(p_k^2)}{d\alpha_1} = 0; \quad k = 1, 2, \ldots n,$$

and similarly

$$\Sigma\alpha_k\frac{d(p_k^2)}{d\alpha_r}=0;\quad k=1,2,\ldots n, \tag{9}$$

if r denotes any one of the numbers from 1 to n.

As mentioned above, α_1 is the $(n-1)!$-fold volume of the $(n-1)$-dimensional n-gon formed by the points $P, e_2, e_3, \ldots e_n$; hence

$$\alpha_1 = p_2 p_3 \ldots p_n \sin\phi_1, \tag{10}$$

where ϕ_1 denotes the $(n-1)$-dimensional angle formed by the $(n-1)$ edges $p_2, p_3, \ldots p_n$, the function $\sin\phi_1$ being defined by formula (10).

Similarly we have in general

$$\alpha_k = p_1 p_2 \ldots p_{k-1} p_{k+1} \ldots p_n \sin\phi_k. \tag{11}$$

Substituting this value for α_k in (9), we have

$$\Sigma p_1 p_2 \ldots p_{k-1} p_{k+1} \ldots p_n \sin\phi_k \frac{d(p_k^2)}{d\alpha_r}=0;\quad k=1,2,\ldots n,$$

or, performing the differentiation and omitting the factor $p_1 \ldots p_n$ which is common to all terms,

$$\Sigma \sin\phi_k \frac{dp_k}{d\alpha_r}=0;\quad k=1,2,\ldots n. \tag{12}$$

We have thus found the general differential equation between the distances of a variable point P from n fixed points $e_1, e_2, \ldots e_n$ in $(n-1)$-dimensional space.

3. In particular we have *for the plane* $(n=3)$

$$\sin\phi_1\frac{dp_1}{d\alpha_r}+\sin\phi_2\frac{dp_2}{d\alpha_r}+\sin\phi_3\frac{dp_3}{d\alpha_r}=0, \tag{13}$$

where ϕ_1, for instance, is the angle between the vectors p_2 and p_3, and where r may be 1 or 2 or 3.

Similarly, *for ordinary space* $(n=4)$ we have:

$$\sin\phi_1\frac{dp_1}{d\alpha_r}+\sin\phi_2\frac{dp_2}{d\alpha_r}+\sin\phi_3\frac{dp_3}{d\alpha_r}+\sin\phi_4\frac{dp_4}{d\alpha_r}=0. \tag{14}$$

Putting $\measuredangle e_r P e_s = \phi_{rs}$, we have for instance

$$\sin\phi_1=\sqrt{1-\cos^2\phi_{23}-\cos^2\phi_{24}-\cos^2\phi_{42}+2\cos\phi_{23}\cos\phi_{24}\cos\phi_{42}}; \tag{15}$$

and substituting this value in the formula

$$a_1 = p_2 p_3 p_4 \sin \phi_1, \tag{16}$$

which follows from (10), we find the well-known expression for six times the volume of the tetrahedron.* The formula (15) expresses the sine of the (three-dimensional) trihedral angle by means of ordinary sine functions.

4. *Minimum of the sum of the distances of the point P from the points* $e_1, e_2, \ldots e_n$. Let us now determine the minimum of the function

$$s = \lambda_1 p_1^u + \lambda_2 p_2^u + \ldots + \lambda_n p_n^u, \tag{17}$$

where u is arbitrary, while $\lambda_1, \lambda_2, \ldots \lambda_n$ may have any positive values. We must have

$$\frac{ds}{d\alpha_1} = \frac{ds}{d\alpha_2} = \ldots = \frac{ds}{d\alpha_n} = 0,$$

or

$$\lambda_1 u p_1^{u-1} \frac{dp_1}{d\alpha_r} + \lambda_2 u p_2^{u-1} \frac{dp_2}{d\alpha_r} + \ldots + \lambda_n u p_n^{u-1} \frac{dp_n}{d\alpha_r} = 0; \qquad r = 1, 2, \ldots n. \tag{18}$$

Now, by (12),

$$\sin \phi_1 \frac{dp_1}{d\alpha_r} + \sin \phi_2 \frac{dp_2}{d\alpha_r} + \ldots + \sin \phi_n \frac{dp_n}{d\alpha_r} = 0; \tag{19}$$

hence we have immediately

$$\lambda_1 p_1^{u-1} : \lambda_2 p_2^{u-1} : \ldots : \lambda_n p_n^{u-1} = \sin \phi_1 : \sin \phi_2 : \ldots : \sin \phi_n. \tag{20}$$

These are, in their most general form, the conditions for a minimum in our problem.

5. In particular, we have *for the plane* ($n = 3$)

$$\lambda_1 p_1^{u-1} : \lambda_2 p_2^{u-1} : \lambda_3 p_3^{u-1} = \sin \phi_1 : \sin \phi_2 : \sin \phi_3, \tag{21}$$

which agrees with the conditions given by Steiner for this case.† Similarly we have *for ordinary space* ($n = 4$)

$$\lambda_1 p_1^{u-1} : \lambda_2 p_2^{u-1} : \lambda_3 p_3^{u-1} : \lambda_4 p_4^{u-1} = \sin \phi_1 : \sin \phi_2 : \sin \phi_3 : \sin \phi_4. \tag{22}$$

* This agrees with von Staudt's definition of the sine of a trihedral angle. See *Crelle's Journal für Mathematik*, vol. 24, p. 255, No. 8.

† STEINER, *Gesam. Werke*, vol. II. p. 17; *Crelle's Journal für Mathematik*, vol. 13, p. 362. Compare the derivation of this special result given by the author in *El progreso matematico*, Zaragoza, vol. 4, p. 174; also K. SIMON, *Ueber den Punkt kleinster Entfernungssumme*, Dissertation, Halle, 1887; UHLICH, *Programmabhandlung*, Grimma 1886; DIPPE, in *Crelle's Journal für Mathematik*, vol. 16.

With $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ and $u = 1$ this reduces to

$$\sin \phi_1 = \sin \phi_2 = \sin \phi_3 = \sin \phi_4. \qquad (23)$$

As these quantities, by (16), measure the trihedral angles formed at P by any three of the vectors p_1, p_2, p_3, p_4, it appears that the condition for the minimum point (as we may call it for the sake of brevity) is the equivalence (as to volume) of these trihedral angles, or the equivalence (as to area) of the triangles cut out by the faces of these angles on a sphere described about P. If the radius of this sphere be taken as 1, the area of each triangle is π.

Let the vectors Pe_1, Pe_2, Pe_3, Pe_4 intersect the sphere in the points P_1, P_2, P_3, P_4. Four such points (by means of which equivalent trihedral angles and spherical triangles can be constructed) may be found by assuming arbitrarily two points, say P_1 and P_2, drawing the diameters P_1P_3 and P_2P_4, and turning the angle P_3PP_4 about its bisector through any angle.* It then appears that the four trihedral angles, as well as the corresponding spherical triangles, are not merely equivalent, but equal.

For, substituting in (23) for the sines their values from (16) and the analogous formulæ, and suppressing the common factor, we find

$$\alpha_1 p_1 = \alpha_2 p_2 = \alpha_3 p_3 = \alpha_4 p_4; \qquad (24)$$

i.e., we have, for instance, $\alpha_3 : \alpha_4 = p_4 : p_3$. But as the tetrahedra α_3, α_4 have the common base Pe_1e_2, their volumes are proportional to the perpendiculars h_4, h_3, let fall from e_4, e_3 on this base; hence

$$\frac{p_4}{p_3} = \frac{h_4}{h_3}, \quad \text{or} \quad \frac{h_3}{p_3} = \frac{h_4}{p_4}.$$

This means that the sines of the angles at which the vectors p_3 and p_4 are inclined to the plane Pe_1e_2, and hence these angles themselves, are equal. It follows that the planes p_3p_4 and p_1p_2 intersect in the bisector of the angle (p_3p_4). As the corresponding relation must hold for the angle (p_1p_2), it appears that the bisectors of the angles (p_3p_4) and (p_1p_2) coincide. For a similar reason the bisectors of (p_1p_3) and (p_2p_4), as well as those of (p_1p_4) and (p_2p_3) must coincide. We have therefore

$$(p_1p_2) = (p_3p_4), \quad (p_1p_3) = (p_2p_4), \quad (p_1p_4) = (p_2p_3);$$

* Comp. STURM, *l. c.* p. 57.

and any two of these angles can, by rotation about the common bisector, be brought into the position of vertical angles, or, conversely, can be generated from this position.

6. If one of the trihedral angles of the given tetrahedron, say e_4, has the value π, the point P will coincide with e_4; the direction of p_4 becomes indeterminate since the vector $p_4 = 0$ and, by (22), $\sin \phi_4 . p_3^{u-1} = 0^0 . \sin \phi_3$, i.e., $\sin \phi_4$ is an indeterminate quantity. It follows that P will coincide with e_4 not only for the value π, but for any greater value of the trihedral angle.

Thus, if the trihedral angle e_4 be $= 2\pi$, the point e_4 lies within the triangle $e_1e_2e_3$. Hence, if one of four points of the same plane lies within the triangle formed by the others, this point is the minimum point for the four points.

If the above-mentioned rotation of the angle P_3PP_4 be equal to zero, the points e_1, e_2, e_3, e_4 form an ordinary quadrilateral in which each of the four points lies outside the triangle of the three others; and P is the intersection of the diagonals and at the same time the minimum point.

If the vertices of the quadrilateral happen to lie in a straight line, say in the order e_1, e_2, e_3, e_4, then the diagonals e_1e_3 and e_2e_4 have in common the vector e_2e_3, instead of a point of intersection, and each point of this vector has the minimum property.

7. With $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ and $u = 2$, (22) reduces to

$$p_1 : p_2 : p_3 : p_4 = \sin \phi_1 : \sin \phi_2 : \sin \phi_3 : \sin \phi_4,$$

or, by (16),

$$p_1 : p_2 : p_3 : p_4 = \frac{\alpha_1}{2p_2p_3p_4} : \frac{\alpha_2}{2p_3p_4p_1} : \frac{\alpha_3}{2p_4p_1p_2} : \frac{\alpha_4}{2p_1p_2p_3},$$

or, finally,

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4; \tag{25}$$

that is, the planes passing through P and the edges of the tetrahedron divide the volume of the tetrahedron into four equal parts. Equation (1) gives in this case

$$P = \frac{e_1 + e_2 + e_3 + e_4}{4}. \tag{26}$$

Either of these properties characterizes the point P as the centroid (barycentre, centre of gravity) of the tetrahedron.*

* GLASER (*Programm, Homburg v. d. H.*, 1887) finds as minimum point the intersection of the lines joining the mid-points of any two opposite edges; this is again the centroid. For a simple proof of this

It retains this property if the four points e_1, e_2, e_3, e_4 lie in the same plane or in the same line.

This result can evidently be extended at once to the case of any n points in a space of $n-1$ dimensions. If the squares of the distances are multiplied by factors $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, it is only necessary, in determining the centroid, to give the points these weights.*

8. *Geometrical interpretations of the general result.*—For the purpose of geometrical interpretation it will be well to remove the sine-functions from formula (20). This will also make it possible to extend the problem to groups of n points situated in a space of less than $(n-1)$ dimensions; the sines in this case would all vanish.

According to Grassmann's geometrical calculus the outer product of two vectors p_2 and p_3, including the angle ϕ_1, is

$$(p_2 p_3) = p_2 p_3 \sin \phi_1.$$

Both members of this equation express also the area (α_1) of the parallelogram determined by the three parts p_2, p_3, ϕ_1.

Similarly, in three-dimensional space, we have for three vectors p_2, p_3, p_4, forming the trihedral angle ϕ_1,

$$\alpha_1 = (p_2 p_3 p_4) = 2 p_2 p_3 p_4 \sin \phi_1$$

as the volume of the parallelepiped determined by the four parts p_2, p_3, p_4, ϕ_1 (comp. formula (16)).

In general we have in an $(n-1)$-dimensional region (comp. formula (10))

$$\alpha_1 = (p_2 p_3 \ldots p_n) = (n-2)!\, p_2 p_3 \ldots p_n \sin \phi_1 \qquad (27)$$

as the $(n-1)$-dimensional volume of the configuration determined by the edges $p_2, p_3, \ldots p_n$ and the $(n-1)$-dimensional angle ϕ_1.

Similarly,

$$\begin{aligned} \alpha_k &= (p_1 p_2 \ldots p_{k-1} p_{k+1} \ldots p_n) \\ &= (n-2)!\, p_1 p_2 \ldots p_{k-1} p_{k+1} \ldots p_n \sin \phi_k. \end{aligned} \qquad (28)$$

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proposition by means of the geometrical calculus see the author's paper *Quelques théorèmes de géométrie à n dimensions*, in *Bull. de la Soc. math. de France*, vol. 10, p. 176.

* This general problem will be found solved in GRASSMANN's *Ausdehnungslehre*, 1862, p. 215.

Hence

$$\sin \phi_k = \frac{(p_1 \dots p_{k-1} p_{k+1} \dots p_n)}{(n-2)!\, p_1 \dots p_{k-1} p_{k+1} \dots p_n}. \quad (29)$$

With the aid of this formula we find from (20), for instance,

$$\frac{\lambda_1 p_1^{n-1}}{\lambda_2 p_2^{n-1}} = \frac{(p_2 p_3 \dots p_n)}{(n-2)!\, p_2 p_3 \dots p_n} : \frac{p_1 p_3 \dots p_n}{(n-2)!\, p_1 p_3 \dots p_n}$$

$$= \frac{(p_2 p_3 \dots p_n)}{(p_1 p_3 \dots p_n)} \cdot \frac{p_1}{p_2},$$

or

$$\frac{\lambda_1 p_1^{n-2}}{\lambda_2 p_2^{n-2}} = \frac{(p_2 p_3 \dots p_n)}{(p_1 p_3 \dots p_n)}.$$

In general, we have

$$\frac{\lambda_r p_r^{n-2}}{\lambda_s p_s^{n-2}} = \frac{(p_1 \dots p_{r-1}\, p_{r+1} \dots p_n)}{(p_1 \dots p_{s-1} p_{s+1} \dots p_n)}; \quad r, s = 1, 2, \dots n;\ r \gtrless s. \quad (30)$$

Thus the sine-functions in (20) are replaced by $(n-1)$-dimensional volumes.

9. The formula just derived is not directly applicable to n points in spaces of less than $(n-1)$ dimensions. But it is closely related to a formula suited for this purpose; and this formula we now proceed to develop.

Let A_r be one of the n fixed points, P a variable point, and O an arbitrary fixed point. We put

$$(O-P) = (\rho), \quad (O-A_r) = (\rho_r), \quad (A_r - P) = (p_r), \quad (31)$$

where the parentheses indicate that the quantities enclosed are vectors, i.e., have not merely length but also direction. By the ordinary laws of vector analysis we have

$$(O-A_r) + (A_r - P) = (O-P), \quad \text{or} \quad (\rho_r) + (p_r) = (\rho),$$

or

$$(p_r) = (\rho) - (\rho_r).$$

If p_r be the numerical value of (p_r), we have

$$p_r = +\sqrt{(p_r)^2},$$

or

$$p_r = +\sqrt{(\rho)^2 - 2[(\rho) \mid (\rho_r)] + (\rho_r)^2}.$$

Hence

$$\frac{dp_r}{d\rho} = \frac{1}{p_r}\left[(\rho) - \frac{[(\rho) \mid (\rho_r)]}{\rho}\right] = \frac{[(\rho) \mid (\rho)] - [(\rho) \mid (\rho_r)]}{p_r\rho}$$
$$= \frac{((\rho) \mid [(\rho) - (\rho_r)])}{p_r\rho} = \frac{[(\rho) \mid (p_r)]}{\rho p_r},$$

or if we omit the parentheses, since in the inner product vectors only can be used,

$$\frac{dp_r}{d\rho} = \frac{(\rho \mid p_r)}{\rho p_r} = \frac{(p_r \mid \rho)}{p_r\rho}. \tag{32}$$

Now, to make the expression

$$s = \lambda_1 p_1^{\,u} + \lambda_2 p_2^{\,u} + \ldots + \lambda_n p_n^{\,u} \tag{17}$$

a minimum, we must have

$$\Sigma\lambda_r p_r^{\,u-1}\frac{dp_r}{d\rho} = 0; \quad r = 1, 2, \ldots n,$$

or

$$\lambda_1 p_1^{\,u-1}\frac{(p_1 \mid \rho)}{p_1\rho} + \lambda_2 p_2^{\,u-1}\frac{(p_2 \mid \rho)}{p_2\rho} + \ldots + \lambda_n p_n^{\,u-1}\frac{(p_n \mid \rho)}{p_n\rho} = 0, \tag{32a}$$

or, finally,

$$\left(\left[\lambda_1 p_1^{\,u-1}\frac{(p_1)}{p} + \lambda_2 p_2^{\,u-1}\frac{(p_2)}{p_2} + \ldots + \lambda_n p_n^{\,u-1}\frac{(p_n)}{p_n}\right] \mid \rho\right) = 0. \tag{33}$$

The meaning of this equation is that the vector enclosed in the square brackets is perpendicular to the vector ρ which joins the arbitrary point O to the minimum point P. As the direction of ρ is arbitrary, equation (33) can evidently hold only if the vector that forms the first factor of the inner product has the value 0. Hence the minimum condition is

$$\lambda_1 p_1^{\,u-1}\frac{(p_1)}{p_1} + \lambda_2 p_2^{\,u-1}\frac{(p_2)}{p_2} + \ldots + \lambda_n p_n^{\,u-1}\frac{(p_n)}{p_n} = 0. \tag{34}$$

Considering that $\frac{(p_r)}{p_r}$ is a unit vector parallel to (p_r), we find our equation to have the following meaning: *On the*

lines PA_1, PA_2, . . . PA_n lay off unit lengths from the minimum point P as origin, increase these lengths in the ratios $1:\lambda_1 p_1^{u-1}$, $1:\lambda_2 p_2^{u-1}$, . . . $1:\lambda_n p_n^{u-1}$, and displace them parallel to themselves so that the initial point of each coincides with the terminal point of the one preceding. The result of this construction will be a closed polygon.

This condition for the minimum point is evidently independent of the number of dimensions of the space in which the points $A_1, A_2, \ldots A_n$ are given. Moreover, as every vector p occurring in the construction has a definite direction, relative minima, such as might arise when one of the terms of the sum should become negative, are excluded.

It should also be noticed that the present method is entirely independent of the preceding investigations. Nevertheless it is easy to transform formula (34) so as to reduce to (30), and thus to establish the required connection between the two methods.

10. Removing the denominators in (34), we find

$$\lambda_1 p_1^{u-1} p_2 p_3 \ldots p_n(p_1) + \lambda_2 p_2^{u-1} p_3 \ldots p_n p_1(p_2) + \ldots + \lambda_n p_n^{u-1} p_1 p_2 \ldots p_{n-1}(p_n) = 0.$$

If this equation be multiplied by the outer product

$$(p_1 p_2 \ldots p_{r-1} p_{r+1} \ldots p_{s-1} p_{s+1} \ldots p_n),$$

all terms excepting those that contain (p_r) and (p_s) will vanish so as to leave

$$\lambda_r p_r^{u-1} p_1 p_2 \ldots p_{r-1} p_{r+1} \ldots p_n (p_r p_1 \ldots p_{r-1} p_{r+1} \ldots p_{s-1} p_{s+1} \ldots p_n) + \lambda_s p_s^{u-1} p_1 p_2 \ldots p_{s-1} p_{s+1} \ldots p_n (p_s p_1 \ldots p_{r-1} p_{r+1} \ldots p_{s-1} p_{s+1} \ldots p_n) = 0,$$

or, omitting the numerical factors common to both terms,

$$\lambda_r p_r^{u-1} p_s (p_1 p_2 \ldots p_{s-1} p_{s+1} \ldots p_n) = \pm \lambda_s p_s^{u-1} p_r (p_1 p_2 \ldots p_{r-1} p_{r+1} \ldots p_n) = 0.$$

The double sign is due to the fact that the introduction of the factors p_r and p_s into their natural places may require an equal or unequal number of changes of sign. As, however, equation (30) is merely a relation between absolute values, we may drop the double sign; the last equation then reduces to the following:

$$\frac{\lambda_r p_r^{u-2}}{\lambda_s p_s^{u-2}} = \frac{(p_1 \ldots p_{r-1} p_{r+1} \ldots p_n)}{(p_1 \ldots p_{s-1} p_{s+1} \ldots p_n)},$$

which agrees with (30).

Now (30) was obtained as merely another form of (20); hence we have thus also established the connection between the n-dimensional generalizations of the solutions formulated by Steiner and Sturm.

11. If one of the given points, say A_n, is to be the minimum point, then $(p_n) = p_n = 0$, hence $\frac{(p_n)}{p_n}$ is indeterminate; and owing to the factor p_n^{u-1} the last term in (34) drops out provided that $u > 1$. If, however, $u = 1$, the factor p_n^{u-1}, and hence the whole last term, is indeterminate; that is, the closing side of the polygon has no determinate value. But we can find a limit for it by the following consideration. According to the laws of the geometrical calculus we have

$$\frac{(p_r \mid \rho)}{p_r \rho} = \cos(p_r \rho). \tag{35}$$

Taking the form (32a) for the minimum condition, we notice first that in the triangle $A_n PO$ the angle $POA_n = 0$ if P coincides with A_n. The sum of the other two angles is π, while either of these angles, taken separately, is indeterminate. Hence $\measuredangle A_n PO$, or $\measuredangle (p_n \rho)$, and consequently $\cos(p_n \rho)$, is indeterminate. By comparing equations (32a) and (34) it will be seen that this cosine has the same value as the closing side of the polygon, which must therefore be less than (or equal to) the unit of length.*

12. With $u = 1$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$, (34) assumes the simple form

$$\frac{(p_1)}{p_1} + \frac{(p_2)}{p_2} + \ldots + \frac{(p_n)}{p_n} = 0, \tag{36}$$

which agrees with the result found by Sturm,† that the vectors (p_1), (p_2), $\ldots$ (p_n) agree in direction with the sides of a closed polygon all of whose sides have the length 1.

With $u = 2$, (34) becomes

$$\lambda_1(p_1) + \lambda_2(p_2) + \ldots + \lambda_n(p_n) = 0; \tag{37}$$

or if the value $(A_r - P)$ be substituted for p_r from (31):

$$P = \frac{\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_n A_n}{\lambda_2 + \lambda_1 + \ldots + \lambda_n}; \tag{38}$$

* This agrees with STURM, *l. c.* p. 52. This special case is designated by Sturm (p. 58) as an *Umartung* of the minimum figure.

† *L. c.* p. 54, No. 5.

that is, P is the centroid of the points $A_1, A_2, \ldots A_n$, these points having the weights $\lambda_1, \lambda_2, \ldots \lambda_n$. (Comp. Art. 7, end.)

13. Substituting the same value of p_r in (34), we find

$$P = \frac{\lambda_1 p_1^{u-2} A_1 + \lambda_2 p_2^{u-2} A_2 + \ldots + \lambda_n p_n^{u-2} A_n}{\lambda_1 p_1^{u-2} + \lambda_2 p_2^{u-2} + \ldots + \lambda_n p_n^{u-2}}; \quad (39)$$

that is: *In a space of any number of dimensions, let $p_1, p_2, \ldots p_n$ be the distances of a variable point P from n fixed points $A_1, A_2, \ldots A_n$; then the function $s = \lambda_1 p_1^u + \lambda_2 p_2^u + \ldots + \lambda_n p_n^u$ has its minimum value when P is taken as the centroid of the points $A_1, A_2, \ldots A_n$, the weights given to these points being $\lambda_1 p_1^{u-2}, \lambda_2 p_2^{u-2}, \ldots \lambda_n p_n^{u-2}$.*

The minimum point has thus been found as a centroid in the most general case.

In particular, with $u = 1$, $n = 3$, and $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we have the corollary: *Let p_1, p_2, p_3 be the distances of the vertices of a triangle from the point P for which the sum of the distances is a minimum; then P is the centroid of the vertices, the weights of these points being $\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}$.*

It further appears from (39) that P can be constructed as centroid only in the case $u = 2$, since in all other cases the weights to be given to the points $A_1, A_2, \ldots A_n$ are functions of the unknown distances $p_1, p_2, \ldots p_n$.

14. *Remarks concerning the construction of the minimum point.*—In the case $u = 2$ the construction of the minimum point reduces, by formula (39), to an ordinary construction of a centroid because the unknown quantities $p_1, p_2, \ldots p_n$ disappear from the right-hand member.

In the case $u = 1$, formula (20) is simplified in a similar way and shows that the sines of the $(n-1)$-dimensional angles formed by joining P to the n given points are in the ratio of the given numbers $\lambda_1, \lambda_2, \ldots \lambda_n$. If, in particular, these numbers are all equal, the corresponding angles are equal, as was known to be the case for $n = 3$ and $n = 4$. To construct the point it is only necessary to satisfy this condition; this problem seems to have been solved only for $n = 3$.

But formula (20), as well as (34), can be used with any value of the numbers $u, \lambda_1, \lambda_2, \ldots \lambda_n$ for the purpose of solving the converse problem: a point P being given in an $(n-1)$-dimensional space, to find a system of n points, for which P is the minimum point satisfying the condition (17). For, let c be any constant vector, subject only to the condition

$c > \lambda_r p_r^{n-1}$ for every one of the values $r = 1, 2, \ldots n$, and let us put, conformably to (20),

$$\frac{\lambda_r p_r^{n-1}}{\sin \phi_r} = c, \quad \text{and hence} \quad \sin \phi_r = \frac{\lambda_r p_r^{n-1}}{c}; \qquad (40)$$

then the following construction results for the required points. From P draw any n half-rays such that no $(n - s)$ rays lie in a space of less than $(n - s)$ dimensions; these rays fully determine the n quantities $\sin \phi_r$. Every point A_r of the required system is now given on its half-ray by means of its distance p_r from P which by (40) becomes

$$p_r = \sqrt[n-1]{\frac{c \sin \phi_r}{\lambda_r}}. \qquad (41)$$

If $n = 1$, the points may have any position on their half-rays.

After thus deducing from (20) the construction, in special cases, of the minimum point for n points in an $(n - 1)$-dimensional space, it remains to show how these results can be extended to any number of points in such a space.

15. This generalization can be effected by first allowing one of the n points to be removed to the $(n-2)$-dimensional space determined by the $(n - 1)$ remaining points, or by projecting the whole group of points on the $(n - 2)$-dimensional space, and by determining the effect of this transformation on the position of the minimum point. Next, 2 points of the new group are removed to the $(n - 3)$-dimensional space determined by the $(n - 2)$ remaining points; and so on. In this way any number of points in a space of any number of dimensions can be investigated.

The groups of points obtained in this way will show interesting spatial relations which affect the position of the minimum point.* The simplest cases may serve for illustration.

(1) Let P be the point the sum of whose distances from *three points in a plane* A_1, A_2, A_3 is a minimum. As A_3 approaches the straight line $A_1 A_2$, the point P will approach A_3, coinciding with it as soon as $\measuredangle A_1 A_3 A_2 = 120°$; and A_3 henceforth remains minimum point even when A_3 comes to lie on the vector $A_1 A_2$.—If A_3 approaches the prolongation of this vector, say on the side of A_1, the point A_1 becomes minimum point as soon as $\measuredangle A_3 A_1 A_2 = 120°$, and remains so even

* As to these relations see the paper by the author: *Ueber die verschiedenen Formen von Gruppen, welche r beliebige Punkte im n-dimensionalen Raume bilden können*, in *Hoppe's Archiv der Math. und Phys.* (2), vol. 10, (1891), p. 292 sq.

when A_3 comes to lie on A_2A_1 produced. Hence, for three points in a line, the one included between the two others is always the minimum point.

(2) Let P be the point for which the sum of the distances from *four points in space* A_1, A_2, A_3, A_4 is a minimum. These points can come to fall into a plane in two different ways: First, it is possible that one of the points, A_4, approaches the area of the triangle $A_1A_2A_3$. Then P will come to coincide with A_4 as soon as the trihedral angle whose vertex is A_4 has become equal to π, and A_4 remains minimum point even when this point falls into the area $A_1A_2A_3$.—Secondly, the edge A_1A_2 may move until it comes to intersect A_3A_4. As the angles A_1PA_2 and A_3PA_4, as well as the angles A_1PA_4 and A_2PA_3, must have the same bisectors, the point P will now come to coincide with the intersection of the diagonals of the quadrilateral $A_1A_2A_3A_4$.

If, moreover, in the *former* of these two cases the points A_1, A_2, A_3 fall into a straight line, while the minimum point A_4 is constantly situated within the triangle $A_1A_2A_3$, the point A_4 itself will fall into this line. But now A_4 is no more the only minimum point; every point of the vector A_2A_4 will have this property. In the *latter* of the above cases, if by turning one diagonal until it coincides with the other, the four points be brought into the same straight line, every point of the shorter diagonal, being a point common to both diagonals, is a minimum point, conformably to the preceding result. (Compare also Art. 6.)

16. The construction of the system $A_1, A_2, \ldots A_n$ for a given minimum point P is, however, effected far more readily by means of the more general formula (34). It is only necessary to construct in $(n-1)$-dimensional space a closed polygon having n, or any greater number of, equal sides, to transfer these sides to parallel positions with the common origin P, and to stretch these vectors in the ratios $1 : \lambda_1 p_1^{n-1}$, $1 : \lambda_2 p_2^{n-1}, \ldots 1 : \lambda_n p_n^{n-1}$. The terminal points of the stretched vectors are the required points.

17. The construction of the point P from given points A_1, $A_2, \ldots A_n$ can be reduced to the solution of an algebraic equation when the points are situated on a straight line.

Let the straight line be determined by the points e_1 and e_2, and let

$$P = \alpha e_1 + (1-\alpha)e_2\,; \quad A_r = \alpha_r e_1 + (1-\alpha_r)e_2. \qquad (42)$$

Then we have

$$(p_r) = (A_r - P) = (\alpha - \alpha_r)(e_1 - e_2),$$

or, denoting the unit of length $(e_1 - e_2)$ by a_r,

$$(p_r) = (\alpha - \alpha_r)(a_r); \quad p_r = \alpha - \alpha_r,$$

whence

$$\frac{(p_r)}{p_r} = (a_r). \tag{43}$$

By the substitution of this value the condition (34) becomes

$$\lambda_1 p_1^{u-1}(a_1) + \lambda_2 p_2^{u-1}(a_2) + \ldots + \lambda_n p_n^{u-1}(a_n) = 0, \tag{43a}$$

or, since the collinear vectors $(a_1), (a_2), \ldots (a_n)$ can be replaced by their numerical value $\pm a$,

$$\pm \lambda_1 p_1^{u-1} a \pm \lambda_2 p_2^{u-1} a \pm \ldots \pm \lambda_n p_n^{u-1} a = 0. \tag{44}$$

Two cases must now be distinguished. (1) *If u be even*, the powers p_r^u in (17) are positive, whether p_r be positive or negative. The same must hold for the derivatives of these powers so that formula (44) can be written

$$\lambda_1 p_1^{u-1} + \lambda_2 p_2^{u-1} + \ldots + \lambda_n p_n^{u-1} = 0,$$

or

$$\lambda_1(\alpha - \alpha_1)^{u-1} + \lambda_2(\alpha - \alpha_2)^{u-1} + \ldots + \lambda_n(\alpha - \alpha_n)^{u-1} = 0. \tag{45}$$

This equation determines the number α on which the position of the point P depends. Thus, for $u = 2$, we find

$$\alpha = \frac{\lambda_1\alpha_1 + \lambda_2\alpha_2 + \ldots + \lambda_n\alpha_n}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}; \tag{46}$$

that is, P is the centroid of the points A_r, the weights being λ_r, as was shown above.

(2) *If u be odd*, we must first determine the interval within which the minimum point is comprised. To do this we let P coincide successively with $A_1, A_2, \ldots A_n$ and determine the value of s in each case. Suppose this value diminishes up to A_r and then increases; by moving the point P to each side of A_r it will appear whether the minimum point lies on $A_{r-1}A_r$ or on A_rA_{r+1}. If, for instance, the latter be the case, α must be found from the equation

$$\lambda_1(\alpha - \alpha_1)^{u-1} + \ldots + \lambda_r(\alpha - \alpha_r)^{u-1} - \lambda_{r+1}(\alpha - \alpha_{r+1})^{u-1} - \ldots - \lambda_n(\alpha - \alpha_n)^{u-1} = 0. \tag{47}$$

18. With $u = 1$ this equation assumes the form

$$\lambda_1 + \ldots + \lambda_r - \lambda_{r+1} - \ldots - \lambda_n = 0. \qquad (48)$$

It may happen that the given values of λ satisfy this equation identically. In this case, P is any point of the vector A_rA_{r+1}. If, in particular, $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$, the equation (48) is satisfied if n be even and $r = \frac{1}{2}n$. Every point of the mid-vector $A_{\frac{n}{2}}A_{\frac{n}{2}+1}$ is then a minimum point.

If the quantities λ do not satisfy (48), it becomes necessary to fall back on (43a), which now assumes the form

$$\lambda_1(a_1) + \lambda_2(a_2) + \ldots + \lambda_n(a_n) = 0. \qquad (49)$$

This equation can be satisfied, independently of the quantities λ, if any one of the quantities (a), say (a_r), assumes the indeterminate form $\frac{0}{0}$; by (43), this requires $p_r = 0$. Hence, in this case, one of the given points, A_r, is minimum point. If, moreover, $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$, and n is odd, we have $r = \frac{1}{2}(n+1)$ and the mid-point of the n points is minimum point. Simple geometrical considerations will confirm these particular results, which have also been mentioned by Mr. Sturm. They are here pointed out in order to call attention to the uniformity of our method (comp. Art. 11).

19. The most general case in which the minimum point becomes a vector is given, according to (39), by the equation

$$\lambda_1 p_1^{u-2} + \lambda_2 p_2^{u-2} + \ldots + \lambda_n p_n^{u-2} = 0.$$

For, in this case, the centroid determined by this equation has the coefficient zero and reduces therefore to a vector, according to the rules of the geometrical calculus.

20. The case $u = 1$ admits a practical application. Let the points $A_1, A_2, \ldots A_n$ be railway stations, arbitrarily distributed along a railway line which need not be straight; and let $\lambda_1, \lambda_2, \ldots \lambda_n$ be numbers measuring their amount of traffic with a central station (for instance, the numbers indicating the population, or any other quantities proportional to the passenger or freight traffic with the central station); then the minimum point P determines that position of the central station for which the total mutual traffic with this central station can be carried on with the least expense of money and time. An opinion can thus be formed as to the best position for erecting a new central station, or as to the relative advantages of one already in existence.*

* On the extension of the minimum problem to a group of points which continuously fill a vector, an area, or a volume, and on its appli-

21. *Application to a problem in mechanics.*—Let n fixed masses $\lambda_1, \lambda_2, \ldots \lambda_n$ be given in an $(n-1)$-dimensional space; and let these masses attract, according to Newton's law, a movable mass μ, situated within the n-gon formed by the masses λ. Where should the mass μ be placed to make the absolute sum of the attractions exerted on it a minimum?

Let $e_1, e_2, \ldots e_n$ be the centroids at which the masses λ are assumed to be concentrated, and let P be determined by the equation (1), as in Art. 2. To satisfy the condition concerning the movability of P it is only necessary to assume $\alpha_1, \alpha_2, \ldots \alpha_n$ all positive. Putting $u = -2$ in the equation of condition (17) and multiplying this equation by μ, we find

$$s\mu = \frac{\lambda_1\mu}{p_1^2} + \frac{\lambda_2\mu}{p_2^2} + \ldots + \frac{\lambda_n\mu}{p_n^2}. \tag{50}$$

as the function that is to become a minimum. Everything else follows by putting $u = -2$ in the results obtained before.

Special Cases. (1) $n = 2$.—In this case, which on account of its simplicity was not mentioned above, we find from (30), by putting $r = 1$, $s = 2$, and from (28), with $\alpha_1 = p_2$, $\alpha_2 = p_1$,

$$\frac{\alpha_1}{\alpha_2} = \frac{\sqrt[3]{\lambda_2}}{\sqrt[3]{\lambda_1}}; \tag{51}$$

that is, the distances of the point of least attraction from the attracting points are to each other as the cube roots of the masses of these points. For it follows from the general equation of the point P (comp. formula (1)), viz.,

$$(\alpha_1 + \alpha_2)P = \alpha_1 e_1 + \alpha_2 e_2,$$

that

$$\frac{\alpha_1}{\alpha_2} = \frac{(e_2 - P)}{(P - e_1)}.$$

(2) $n = 3$.—Formula (20) gives the conditions

$$\lambda_1 p_1^{-3} : \lambda_2 p_2^{-3} : \lambda_3 p_3^{-3} = \sin\phi_1 : \sin\phi_2 : \sin\phi_3,$$

or

$$p_1 : p_2 : p_3 = \sqrt[3]{\frac{\lambda_1}{\sin\phi_1}} : \sqrt[3]{\frac{\lambda_2}{\sin\phi_2}} : \sqrt[3]{\frac{\lambda_3}{\sin\phi_3}}, \tag{52}$$

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cation to the determination of the most advantageous position of the farmhouses on a farm of given shape and area, see the paper by the author *Sur les distances moyennes entre un point et des variétés de points, discrètes ou continues*, in *Comptes rendus de l'Association française pour l'avancement des sciences*, 1887, p. 266, sq., especially p. 275.

or, if the sine functions be eliminated by means of (11),

$$\alpha_1 : \alpha_2 : \alpha_3 = \lambda_1(p_2p_3)^4 : \lambda_2(p_3p_1)^4 : \lambda_3(p_1p_2)^4. \qquad (53)$$

While (52) does not enable us to construct the point of least attraction, it furnishes a solution of the converse problem: to determine the ratios of the masses of three points so as to make the sum of their attractions on a point P within their triangle a minimum.

If, in (50), we put $n = 2$ and $\alpha_1 + \alpha_2 = 1$, and hence $p_1 + p_2 = 1$, this equation can be regarded as that of a curve whose ordinate s represents the sum of the attractions exerted by the points e_1 and e_2 on the foot of the ordinate. This curve approaches asymptotically the perpendiculars erected on the vector $(e_1 - e_2)$ at e_1 and e_2; and the point of minimum attraction corresponds to its lowest point. Similarly, in the case $n = 3$, the sum of the attractions exerted by the vertices of the triangle on any point within this triangle can be represented by the ordinate of a surface, erected at this point at right angles to the plane of the triangle. This suggestion may here suffice.

22. *Concluding remark.*—Further results concerning generalizations of the problem of the minimum sum of distances are reserved for a future communication.

WAS THE BINOMIAL THEOREM ENGRAVEN ON NEWTON'S MONUMENT?

BY PROFESSOR FLORIAN CAJORI.

Moritz Cantor, in a recently published part of his admirable work, *Vorlesungen über Geschichte der Mathematik*, speaks of the "Binomialreihe, welcher man 1727 bei Newtons Tode . . . eine solche Wichtigkeit beilegte, dass man sie aus allen anderen Leistungen des Vestorbenen auswählte, um als Inschrift auf seinen Grabstein in der Westminsterabtei eingemeisselt zu werden" (vol. III, p. 65). In my own brief *History of Mathematics*, p. 218, I say that it is *not* true that the binomial theorem is engraved on Newton's monument.

The above passage in Cantor's work leads us to re-examine the subject. The first step naturally is to question the monument, but we do so in vain. Says Dr. Granville, the present Dean of Westminster, in a letter to the writer: "In front of the half-recumbent figure of Sir Isaac Newton are two winged youths holding a small scroll in which are still, according to Neale, some mathematical figures. . . . I fear that the figures on the small marble scroll are *quite obliterated*. A mathemat-

ical friend mounted the monument for me. Time, I fear, and London atmosphere have done their sad work, and the older guide-books, Dart and Neale, while they carefully copy other inscriptions, naturally do not preserve for us mathematical formulæ." A friend of the writer of this article reports, "I looked as close as a chair would permit and saw no sign of engraving on the scroll."

Our American cyclopædias, almost without exception, assert that the Binomial Formula is engraven on Newton's tomb or his monument. This assertion is supported by an author who wrote only sixteen years after Newton's death. In the *New Mathematical Dictionary* by E. Stone, F.R.S., Second Edition, London, 1743, under "Binomial Root," we read, "This admirable theorem, which is put upon his monument in Westminster Abbey, has never been yet demonstrated."

Had the inscription ever been made, then the general interest attached to it would tempt mathematical writers and biographers to speak of it. Now the very fact that all authoritative writers on Newton, including those most likely to know from personal observation, are silent on this subject (though they frequently take pains to copy the Latin inscriptions), leads us to the conclusion that the theorem was never engraven. This inference has been drawn after the examination (either by the writer or by friends) of the following works: "Life of Newton" in the Biographia Britannica (1760), Maclaurin's Exposition of Sir Isaac Newton's Philosophical Discoveries (1775), Maclaurin's Treatise of Fluxions (1742), Maclaurin's Treatise of Algebra (1779), Horsley's edition of Newton's works (1779–85), Saunderson's Algebra (1740), Hutton's Philosophical and Mathematical Dictionary (1796), Biot's article "Newton" in Biographie Universelle (1822), Arago's Notices Biographiques (1855), Life of Newton in the Library of Useful Knowledge (1833), the 8th and 9th editions of the Encyclopædia Britannica, the histories of mathematics by Montucla, Bossut, Arneth, Ball, Marie, Hoefer, and Fink.

While guide-books naturally would not copy formulæ, we should rather expect the old guides of Westminster Abbey to mention the name "Binomial Theorem," had the theorem been inscribed on Newton's tomb or monument.

Dean Stanley, in his *Historical Memorials of Westminster Abbey*, 1876, says nothing of the binomial theorem, though he quotes even passages which were *proposed* for inscription upon the monument and tomb, but which were rejected. "On the gravestone (restored to its place in 1866) are written the words which here acquire a significance of more than usual solemnity—'Hic depositum quod mortale fuit Isaaci Newtoni'" (p. 314).

David Brewster, in his *Life of Sir Isaac Newton*, 1831,

says, "On the sarcophagus is placed the figure of Sir Isaac Newton in a cumbent posture, with his elbow resting on several of his works. Two youths stand before him with a scroll, on which is drawn a remarkable diagram relative to the solar system, and above that is a converging series." Brewster would surely have said "binomial theorem" instead of "converging series," had the theorem been there.

In the article "Newton" in the *Penny Cyclopædia* (1840) we read, "It is not true that the binomial theorem is also engraven upon it [the monument], though it is so stated under 'Binomial Theorem,' on the authority of several writers." Now all mathematical articles in this cyclopædia are De Morgan's. From the numerous cross-references to mathematical articles made under "Newton" and the absence of references to other articles (such as "Light" and "Gravitation," not written by De Morgan), as well as from other considerations, we infer that the article "Newton" is De Morgan's also. Remembering his accuracy in details, much weight must be attached to this denial. The above passage is reprinted in the *English Cyclopædia* (with the omission of three words), while under "Binomial Theorem" we read, "It is often said, but wrongly, to have been engraven on his tomb in Westminster Abbey."

Thus it appears pretty conclusively that there is no more foundation for the statement that the Binomial Theorem was inscribed on Newton's tomb or monument than there was authority for the story of the "apple" or for the use of the word *fluxion* and the notation $\dot{x}$ by all English writers previous to 1704 (excepting Newton and Cheyne) and by Stone in 1743, in the sense of an infinitely small increment.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held Saturday afternoon, October 27, at three o'clock, the President, Dr. McClintock, in the chair. The following persons, having been duly nominated and being recommended by the Council, were elected to membership: Miss Charlotte Cynthia Barnum, New Haven, Conn.; Professor Robert Lee Flowers, Trinity College, Durham, N. C.; Mr. George H. Hallett, University of Pennsylvania, Philadelphia, Pa.; Mr. Edgar Odell Lovett, University of Virginia, Charlottesville, Va.; Mr. Elmer A. Lyman, University of Michigan, Ann Arbor, Mich.; Mr. Max Osterberg, Columbia College, New York, N. Y.; Dr. James P. Pierpont, Yale University, New Haven, Conn.; Mr. Ralph Augustus Roberts, New York, N. Y.

The following papers were read:

(1) "A method for calculating simultaneously all the roots of an equation—Second paper," by Dr. Emory McClintock.

(2) "Determination in a literal form of the motion of the Moon's perigee," by Dr. G. W. Hill.

Dr. McClintock's paper was a continuation of a previous paper (see BULLETIN for October, 1894, pp. 2–4). It will be printed in the *American Journal of Mathematics*.

Dr. Hill's paper will appear in the *Astronomical Journal*. He has furnished the following abstract: "The earlier investigators of the lunar theory contented themselves with giving a numerical result for the motion of the Moon's perigee; and Plana was the first to derive a literal expansion. Considering only the principal part, he carried it to the term in m^7 inclusively, m denoting the ratio of the month to the year. This expression was afterwards confirmed by Pontécoulant and others. Delaunay in elaborating his lunar theory added the terms in m^8 and m^9. Comparing these terms with the numerical result of my paper on the motion of the Moon's perigee of 1877, it seemed to me that they were probably somewhat in error. However, as this was not certainly proved, I abstained from making any assertion to this effect. But Mr. Andoyer has, within a few years, given values for the coefficients of these two terms differing somewhat from those of Delaunay, and which he claims ought to be substituted for the latter. It seemed important that these new values of Mr. Andoyer should be subjected to verification. This, accordingly, I have done in the present paper, with the result that Mr. Andoyer is right. To give more interest to the paper I have determined the two next following terms of the series, viz., those in m^{10} and m^{11}."

AMONG the courses announced for the winter semester at the University of Göttingen are the following: Klein, Theory of numbers—Weber, Introduction to the higher mathematics, Calculus of variations—Schönflies, Algebra, Descriptive geometry—Burkhardt, Axioms of geometry—Ritter, Introduction to the theory of functions—Pockels, Hydrodynamics—Schering, Partial differential equations—Riecke, Thermodynamics—Voigt, Applications of potential theory—Ambronn, Higher geodesy—Schur, Determination of orbits of planets and comets.

THE fall announcements of Macmillan & Co. include a long series of works. We may mention: "A Treatise on Bessel's Functions," by G. B. Mathews and A. Gray; "Elliptic Functions," by A. C. Dixon; "Practical Plane Geometry," by J. H. Spanton; "An Introductory Account of Certain Modern Ideas

and Methods in Plane Analytical Geometry," by Charlotte A. Scott; "An Elementary Treatise on the Theory of Functions," by J. Harkness and F. Morley; "Integral Calculus and Differential Equations for Beginners," by Joseph Edwards; "Geometrical Conic Sections," by Charles Smith; "Elementary Mensuration," by F. H. Stevens; "The Theory of Light," by Thomas Preston, second edition, thoroughly revised; "Magnetism and Electricity," by Andrew Gray.

WORKS of interest announced by the Cambridge University Press are: "The Scientific Papers of John Couch Adams"; "A Treatise on Spherical Astronomy," by Prof. Sir Robert S. Ball; "Hydrodynamics: a Treatise on the Mathematical Theory of the Motion of Fluids," by Prof. H. Lamb, new edition; "Catalogue of Scientific Papers Compiled by the Royal Society of London," new series for the years 1874–83, vol. XI, in the press; "A Treatise on Geometrical Optics," by R. A. Herman; "An Introduction to Abel's Theorem and the Allied Theory," by H. F. Baker; "A Treatise on Geometrical Conics," by F. S. Macaulay; "Euclid's Elements of Geometry, Books XI and XII," by H. M. Taylor; "Mechanics and Hydrostatics" and "Electricity and Magnetism," both by R. T. Glazebrook.

Sampson Low & Co.'s announcements include "A Textbook of Mechanics and Hydrostatics," by Herbert Hancock, and "Thermodynamics: treated with elementary mathematics, and containing applications to animal and vegetable life, tidal friction, and electricity," by J. Parker. In Mr. Murray's list is a "History of Astronomy," by Arthur Berry; and in Chapman & Hall's a "Practical Plane and Solid Geometry," by Henry Angel. T. S. F.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BÖKLE (C.). Die Ordnungskurven des Nullsystems. Ihre Verteilung im Raum und ihre Konstruction. [Progr.] Frankenthal, 1894. 8vo. 26 pp. Illustrated.

BOREL (E.) et DRACH (J.). Introduction à l'étude de la théorie des nombres et de l'algèbre supérieure. D'après des conférences faites à l'École normale supérieure par J. Tannery. Paris, Nony, 1895. 8vo. 9 and 355 pp.

CAYLEY (A.). Collected mathematical papers, Vol. VII. Cambridge, University Press, 1894. 4to. 25s

GLAUNER (T.). Ueber den Verlauf der Potentialfunctionen im Raume. Leipzig, 1894. 8vo. 62 pp. Mk. 1.20

HANCOCK (H.). Eine Form des Additionstheorems für hyperelliptische Functionen erster Ordnung. Berlin, 1894. 4to. 43 pp. Mk. 1.80

HARNACK (A.). Introduction to the elements of the differential and integral calculus. Translated by G. L. Cathcart. London, 1894. 8vo. Cloth. 10s. 6d.

HIPPARCHUS BITHYNUS. In Arati et Eudoxi Phaenomena commentariorum libri tres. Ad codicum fidem recensuit et germana interpretatione instruxit C. Manitius. Lipsiae, 1894. 8vo. 34 and 376 pp. Mk. 4.00

KRUG (A.). Invarianten-Theorie der linearen Differentialgleichungen und Auflösung der algebraischen Gleichungen vierten Grades. [Progr.] Prag, 1894. 8vo. 24 pp.

KVITEK (A.). Příspevek ku zobrazování os trojosého ellipsoidu, daného třemi sdruženými průměry. II. Kterak strojiti lze normaly ellipsy z bodu mimo křivku v případu obecném w některých případech zvlástních. [Progr.] Brné, 1894. 8vo. 15 pp. Illustrated.

LASALA Y MARTINEZ (D. A.). Teoria de las cantidades imaginarias. Bilbao, 1894. 8 and 149 pp. Fr. 6.00

MARTONE (M.). Sulla somma delle potenze simili della serie dei numeri naturali. Catanzaro, Asturi, 1894. 4to. 18 pp.

OTTE (A.). Grenzbereiche und Flächenstücke kleinsten Flächeninhalts der gewöhnlichen Schraubenfläche. [Diss.] Göttingen, 1894. 8vo. 32 pp.

PICARD (É.). Traité d'analyse (cours de la Faculté des sciences). [En 4 volumes.] Tome III: Des singularités des integrales des équations différentielles; étude du cas où la variable reste réelle. Paris, Gauthier-Villars, 1894. 8vo. Fr. 14.00

SCOTT (C. A.). An introductory account of certain modern ideas and methods in plane analytical geometry. London and New York, Macmillan, 1894. 8vo. 288 pp. Cloth. $2.50

SERVAIS (C.). Sur les imaginaires en géométrie. Applications à la théorie des cubiques gauches. Gand, 1894. 8vo. 64 pp. Mk. 1.80

WENDT (E.). Arithmetische Studien über den "letzten" Fermat'schen Satz, welcher aussagt, dass die Gleichung $a^n = b^n + c^n$ für $n > 2$ in ganzen Zahlen nicht auflösbar ist. [Diss.] Berlin, 1894. 4to. 17 pp.

II. ELEMENTARY MATHEMATICS.

ALGEBRA. Specific subject tests. Stage I: Answers. London, Simpkin, 1894. 12mo. 4d.

BOCKWOLDT (G.). Die analytische Geometrie in der Prima des Gymnasiums. Theil I. Neustadt, 1894. 8vo. 16 pp. 4 plates.

CARBONI (S. O.). Geometria descrittiva ed elementare ed alcune sue applicazioni. Vol. I. Torino, Paravia, 1894. 13 and 138 pp. Fr. 2.75

DUPUIS (J.). Tables de logarithmes à cinq décimales, d'après J. de Lalande, disposées à double entrée et revues par J. Dupuis. Edition stéréotype, contenant les logarithmes des nombres de 1 à 10,000, les logarithmes des sinus et des tangentes des angles calculés de minute en minute jusqu'à 90 degrés, plusieurs tables usuelles et un grand nombre de formules et de nombres utiles. Paris, Hachette, 1894. 16mo. 4 and 230 pp. Fr. 2.00

HOLZMÜLLER (G.). Methodisches Lehrbuch der Elementar-Mathematik, im Anschluss an die neuen Lehrpläne. Theil II. Leipzig, 1894. 8vo. 7 and 273 pp. Illustrated. Mk. 3.00

KOENIG (M.). Die geometrische Theilung des Winkels. Berlin. 1894. 8vo. 32 pp. Mk. 2.00

MILNE (W. J.) Elements of algebra. A course for grammar schools and beginners in public and private schools. New York, American Book Co. [1894]. 8vo. 200 pp. Half leather. $0.60

PETR (K.). Ulohy trigonometrické. [Progr.] Jicin, 1894. 8vo. 24 pp.

REUM (A.). Der mathematische Lehrstoff für den Quartaner der höheren Lehranstalten in entwickelnder Lehrweise bearbeitet. Barmen-Wupperfeld, 1894. 8vo. 35 pp. Illustrated. Mk. 1.50

SCHWERING (K.) und KRIMPHOFF (W.). Anfangsgründe der ebenen Geometrie, nach den neuen Lehrplänen bearbeitet. Freiburg, 1894. 8vo. 8 and 132 pp. Illustrated. Mk. 1.80

SONNET (H.) Primeros elementos de geometria, con las principales aplicaciones al dibujo lineal, al levantamiento de planos, a la agrimensura, etc., etc.; traducidos del francés por J. Frontera. Cuarta edición. Paris, Hachette, 1894. 16mo. 239 pp. Fr. 3.00

TOMBECK (H. E.). Traité de géométrie élémentaire, à l'usage des classes de sciences des lycées et des candidats au baccalauréat ès sciences et aux écoles du gouvernement, avec de nombreux exercices; 15e édition. Paris, Hachette, 1894. 8vo. 504 pp. Fr. 5.00

III. APPLIED MATHEMATICS.

BACH (C.). Versuche über die Widerstandsfähigkeit von Kesselwandungen. Heft II: Die Berechnung flacher, durch Anker oder Stehlbolzen unterstützter Kesselwandungen und die Ergebnisse der neuesten hierauf bezüglichen Versuche. Berlin, 1893. 4to. 56 pp. Illustrated. Mk. 3.00

BAZALA (J.). Beleuchtungs-Constructionen für windschiefe Flächen mit einer Richtebene. [Progr.] Bielitz, 1894. 8vo. 16 pp.

BOUQUET (R. P.) et LOPPÉ. Traité théorique et pratique des courants alternatifs industriels. (En 2 volumes.) Vol. I. Paris, 1894. 8vo. 288 pp. Fr. 10.00

BRIGGS (W.) and BRYAN (G. H.). Elementary text-book of mechanics. Part II: Statics. London, 1893. 8vo. 220 pp. Cloth. 1s. 6d.

CORONAT (G.). Du mécanisme de l'accommodation de l'œil. [Thèse.] Lyon, Rey, 1894. 4to. 83 pp.

FACCIOLI (A.). Teoria del volo e della navigazione aerea. Ricerche sperimentali sulla resistenza dell'aria; teoria dell'elice e del timone. Milano, 1894. 8vo. 318 pp. Fr. 7.50

FEGERL (J.). Ableitung der Schwingungsdauer des mathematischen Pendels. [Progr.] Ostrau, 1894. 8vo. 9 pp.

FIDLER (J.). Die Elemente der Centralbewegung in Kegelschnitten. [Progr.] Arnau, 1894. 8vo. 30 pp.

GIANTURCO (G.). Teorie matematiche della ricerca degli elementi compensati per la determinazione dei punti trigonometrici catastali in Italia. Potenza, Spera, 1894. 4to. 35 pp.

HANCOCK (H.). Text-book of mechanics and hydrostatics. London, 1894. 8vo. Cloth. 5s.

MOURET (G.). Troisième étude sur l'énergétique: démonstration du principe de l'équivalence. Paris, Niort, 1894. 8vo. 13 pp. Fr. 0.75

MÜHLE (G.). Ueber die Bewegung eines Punktes in einer Ebene, die senkrecht steht auf der Verbindungslinie zweier beliebig anziehenden festen Centren. [Diss.] Halle, 1894. 8vo. 50 pp.

PARKER (J.). Thermo-Dynamics, treated with elementary mathematics and containing applications to animal and vegetable life, tidal friction and electricity. London, 1894. 8vo. Cloth. 10s. 6d.

PAULSEN (A.). Lærebog i Optik. 2., forandrede oplag. Kjöbenhavn, 1894. 8vo. 174 pp. Illustrated. Mk. 3.80

POLLARD (J.) et DUDEBOUT (A.). Architecture navale; théorie du navire. Vol. IV (et dernier): Dynamique du navire; mouvement rectiligne horizontal oblique; mouvement curviligne horizontal; propulsion; vibration des coques des navires à hélice. Paris, Gauthier-Villars, 1894. 8vo. Illustrated. 1 plate. Fr. 13.00

PRESTON (E. D.). Mean Density of the Earth. [Philosophical Society of Washington. Bulletin, Vol. XII.] Washington, 1894. 8vo. pp. 369-396. 4 plates.

PRESTON (S. T.). Ueber das gegenseitige Verhältniss einiger zur dynamischen Erklärung der Gravitation aufgestellten Hypothesen. Leipzig, 1894. 8vo. 20 pp. Mk. 0.80

THORNTON (A.). Theoretical mechanics: Solids, including kinematics, statics, and kinetics. London and New York, Longmans, 1894. 8vo. 422 pp. (Advanced science manuals.) 4s. 6d.

TOMBECK (H. E.) Traité élémentaire de mécanique, à l'usage des classes de sciences des lycées et des candidats au baccalauréat ès sciences et aux écoles du gouvernement. 4e édition. Paris, Hachette, 1895. 8vo. 120 pp. Fr. 2.00

VOLDERAUER (L.). Constructive Behandlung astronomischer Aufgaben (1. Fortsetzung.) [Progr.] Trautenau, 1894. 8vo. 16 pp.